

SOLUTION OF INVERSE PROBLEMS OF HEAT
CONDUCTION BY THE METHOD OF DYNAMIC FILTRATION

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The applicability of Kalman's filter equations to the solution of inverse boundary problems of heat conduction is investigated.

Among the various approaches to the solution of incorrectly stated inverse problems of heat conduction (IPHC), one can distinguish methods unrelated to restrictions of stability on the quantization step of the initial differential statement of the problem and methods in which stability is achieved through a certain choice of the quantization intervals.

The first group [1, 2] includes algorithms based on α -parametric regularization by Tikhonov's method of extremal statement of IPHC, as well as algorithms of iterative regularization. In application to boundary IPHC the latter includes direct approximately analytical and numerical methods employing the principle of stepwise regularization [1].

The choice of the means of solving practical problems should be based on the consideration of a whole series of factors allowing for the special features of the statements of the problems, the complexity of the algorithms, the accuracy of the results, the expenditures of computer time, etc. Such an analysis allows one to bring out and formulate the conditions of rational practical application of the various methods of solving IPHC. Below we will discuss some of these questions for the solution of the inverse problem using equations of dynamic filtration, including a comparison with finite-difference algorithms of the second group.

Kalman's method of optimum dynamic filtration [3, 4] has recently been used rather widely for the solution of inverse problems of heat conduction [5, 6]. This can evidently be explained by the fact that this method provides a sufficiently economical recurrent scheme of solution of a problem, when one successively determines the current estimates of the state and parameters of a thermal model, doing so with allowance for random errors at the input to the system and in the measurements. Moreover, the filter possesses a definite interference protection. The original form of Kalman's filter does not directly generate regularization algorithms, however. Therefore, it is interesting to investigate the possibility of obtaining stable results using an algorithm of optimum dynamic filtration based on the well-known property of stepwise regularization. The present report is devoted to just this question in application to the solution of inverse boundary problems of heat conduction.

The numerical solution of IPHC implies the approximation of the heat-conduction equation by a difference analog, in which case the working stability of the algorithm of dynamic filtration starts to depend significantly on the approximation parameters. The step $\Delta Fo = a\Delta\tau/b^2$ in the Fourier number can serve as the parameter on which the stability of algorithms used for the solution of boundary IPHC depends. One can assume that for a dynamic filtration algorithm there is some critical value ΔFo_{cr} of step, and for smaller steps ($\Delta Fo < \Delta Fo_{cr}$) the distorting influence of the instability properties of the problem will show up. We set the goal of verifying this hypothesis, using the method of practical analysis of correctness, and finding estimates of ΔFo_{cr} for a sufficiently popular statement of IPHC in practice.

Let the temperature at one boundary of a one-dimensional body be reconstructed from temperature measurements at the opposite, thermally insulated boundary. For this problem we adopt the mathematical model

$$\frac{\partial T}{\partial \tau} = a \frac{\partial^2 T}{\partial x^2}, \quad 0 < \tau < \tau_m, \quad 0 < x < b, \quad (1)$$

$$T(x, 0) = 0, \quad 0 \leq x \leq b; \quad (2)$$

$$T(b, \tau) = \varphi(\tau), \quad \frac{\partial T(b, \tau)}{\partial x} = 0, \quad 0 \leq \tau \leq \tau_m, \quad (3)$$

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under the assumption that the initial distribution (2) and the Cauchy conditions (3) and consistent with each other. It is required to reconstruct the temperature $T(0, \tau)$ from measurements of the temperature $T(b, \tau)$. We will take the thermophysical characteristics of the body as exactly known.

We replace the differential statement of the corresponding direct problem by an approximate difference analog corresponding to an implicit scheme of approximation of the heat-conduction equation (1),

$$\frac{T_i^{j+1} - T_i^j}{\Delta\tau} = a \frac{T_{i-1}^{j+1} - 2T_i^{j+1} + T_{i+1}^{j+1}}{\Delta x^2}, \quad i = 2, 3, \dots, N-1, \quad (4)$$

$$j = 1, 2, \dots, M,$$

with discrete boundary conditions

$$T_N^{j+1} = \varphi(\Delta\tau(j+1)), \quad \frac{T_N^{j+1} - T_N^j}{\Delta\tau} = 2a \frac{T_{N-1}^{j+1} - T_N^{j+1}}{\Delta x^2},$$

where $\Delta\tau = \tau_m/M$ and $\Delta x = b/N$ are the steps of the grid in time and in space. In vector notation the finite-dimensional model of the problem has the form

$$\Phi \bar{T}^{j+1} = \bar{T}^j, \quad (5)$$

where Φ is a three-diagonal matrix of size $N \times N$; \bar{T} is the vector of temperatures at nodes of the grid.

Numerical solutions of the direct problem in the range of $\Delta Fo = 0.1-0.0001$ were preliminarily investigated to determine the influence of the approximation parameters on the convergence of the results. These calculations were made for the case when the unknown function was assigned in the form $T(0, \tau) = \sin(\pi\tau/\tau_m)$ and $M = 20$. For a constant ΔFo the numerical solution of the direct problem depends only on the number of nodes over the spatial coordinate, since

$$\frac{a\Delta\tau}{\Delta x^2} = \frac{a\Delta\tau}{(b/N)^2} = \Delta Fo N^2 = S, \quad (6)$$

i.e., the form and order of the three-diagonal matrix Φ are conserved for constant values of ΔFo and N . The results of the calculations for different ΔFo and N are presented in Fig. 1.

To compare the results of solutions at two adjacent grids, we chose the metric

$$\delta(T(N_n), T(N_{n+1})) = \frac{|T_{N_n}^M - T_{N_{n+1}}^M|}{T_{N_n}^M}, \quad (7)$$

where N_n is the number of nodes over the spatial coordinate for the n -th grid; n is the serial number of the grid.

The results of the comparison of solutions in this matrix are presented in Fig. 2a. In Fig. 2b we show the results of a comparison of solutions using a deviation of the type

$$\delta^*(T(N_n), T(N_{n+1})) = \frac{\delta(T(N_n), T(N_{n+1}))}{S_{n+1} - S_n}, \quad (8)$$

where $S_n = \Delta Fo N_n^2$.

It is seen from Fig. 2a that with an increase in N the solutions almost cease to differ from each other: $\Delta Fo \approx 10^{-1}$ for $N \geq 15$, $\Delta Fo \approx 10^{-2}$ for $N \geq 50$, $\Delta Fo \approx 10^{-3}$ for $N \geq 200$, and $\Delta Fo \approx 10^{-4}$ for $N \geq 800$. From this, one can calculate the required number of subdivisions over the spatial coordinates. If a smaller N is chosen, then there is an actual increase in ΔFo . For example, in solving the direct problem larger values of the temperature T_N^1 were obtained for $\Delta Fo = 10^{-4}$ and $N = 5$ than for $\Delta Fo = 10^{-3}$ and $N = 20$.

In solving IPHC by Kalman's filtration method we used the equations

$$\Phi \bar{T}_-^{j+1} = \bar{T}_+^j, \quad (9)$$

$$\Phi P_-^{j+1} \Phi^T = P_+^j, \quad (10)$$

$$K = P_-^{j+1} H^T (H P_-^{j+1} - H^T + r)^{-1}, \quad (11)$$

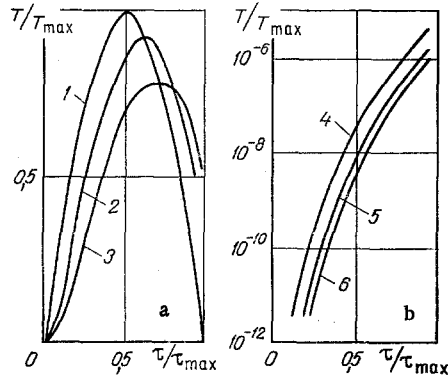


Fig. 1. Results of solution of the direct problem of heat conduction: a) $\Delta Fo = 10^{-1}$; 1) $\varphi(\tau)$; 2) $T_N(\tau)$; $N = 5$; 3) 25; b) $\Delta Fo = 10^{-3}$; 4) 20; 5) 100; 6) 284.

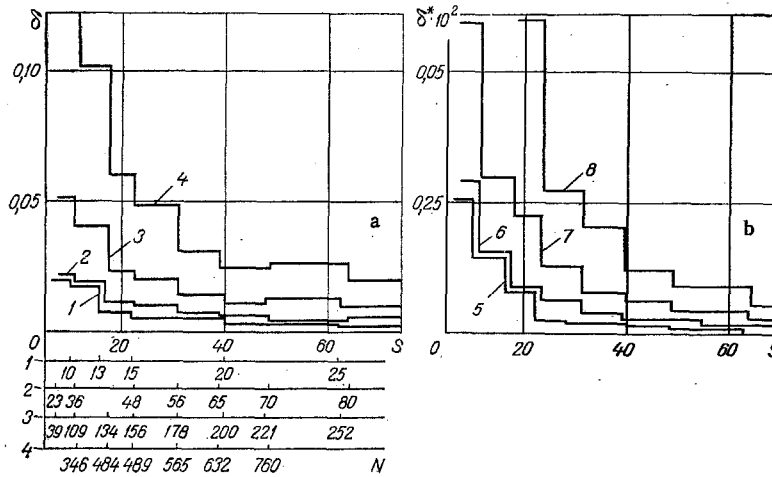


Fig. 2. Results of the comparison of solutions on different grids: 1, 5) $\Delta Fo = 10^{-1}$; 2, 6) 10^{-2} ; 3, 7) 10^{-3} ; 4, 8) 10^{-4} .

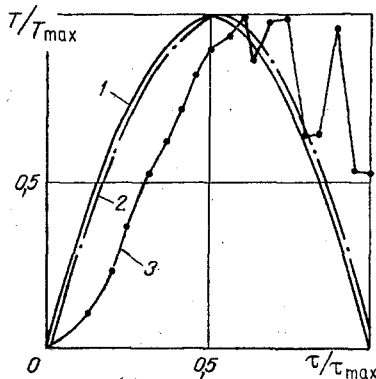


Fig. 3. Results of solution of the inverse problem of heat conduction by Kalman's filtration method: 1) $\varphi(\tau)$; 2) $\Delta Fo = 10^{-2}$; 3) 10^{-3} .

$$\bar{T}_+^{j+1} = T_-^{j+1} + K(\varphi(\Delta\tau(j+1)) - H\bar{T}_-^{j+1}), \quad (12)$$

$$P_+^{j+1} = (E - KH)P_-^{j+1}, \quad (13)$$

where T_{-}^{j+1} and \bar{T}_{+}^{j+1} are the a priori and a posteriori estimates of the temperature vector at the $(j + 1)$ -th moment; P_{-}^{j+1} and P_{+}^{j+1} , a priori and a posteriori values of the covariation matrix of errors of the estimates; r , dispersion of the measurement errors; E , a unit matrix; K , amplification ratio of the filter; H , matrix of the measurement vector, in the given case $H = (0, 0, \dots, 0, 1)$, with a dimensionality $(1 \times N)$; $\varphi(\Delta\tau(j + 1))$ are the results of the measurements obtained at the $(j + 1)$ -th moment.

In the algorithm developed for the solution of IPHC the matrix Φ was not inverted, but the a priori values of the temperature vector T_{-}^{j+1} and the covariation matrix P_{-}^{j+1} were obtained from Eqs. (9) and (10) by the trial-run method. In order that, in estimating the boundary temperature $T(0, \tau)$ of the body, the temperature T_1^j follow its variation, in the filter equations one must maintain the amplification ratio at a high enough level, which was achieved by multiplying the elements along the diagonal of the matrix P_{+}^{j+1} by some experimentally determined number.

The results of the calculation of the direct problem served as the initial data for the solution of the inverse problem. The initial data were not distorted artificially, and only computer rounding errors were present. In the calculation on an ES-1033 computer with an ordinary degree of accuracy seven significant figures were retained in the mantissa of the number, i.e., the rounding errors have the order of 0.000001%. The calculations showed that under these conditions for $\Delta Fo \leq 10^{-3}$ the graphs of the solution of the IPHC acquire an oscillatory form, with the oscillations increasing as ΔFo decreases (Fig. 3). For $\Delta Fo \geq 10^{-2}$ under the same conditions the solution of the inverse problem of heat conduction coincides sufficiently well with the reference solution (Fig. 3).

The following conclusions can be drawn from the results of this work.

1. The method of solving IPHC based on dynamic filtration equations with small values of ΔFo leads to oscillating solutions far from the solution being sought.
2. By applying the method of stepwise regularization one can obtain stable and sufficiently precise results if the choice of the time interval from the condition of the required smoothness of the solution ($\Delta Fo > \Delta Fo_{cr}$) is not at variance with the choice of this step from the condition of obtaining the fine structural features of the function being reconstructed.
3. It is appropriate to use the method of dynamic filtration to solve IPHC for $\Delta Fo \geq 10^{-2}$. In comparing the critical values of the step ΔFo_{cr} obtained in this analysis for the method of solving IPHC using Kalman's filter and the values of ΔFo_{cr} corresponding to direct numerical methods of solving the inverse problem [1] one can conclude that at this level the filtration method has no special advantages over the simpler and more economical determinate numerical methods. Thus, for a difference algorithm for reconstructing the heat flux density based on an implicit scheme of approximation of the inverse boundary problem, an estimate of the critical step gives $\Delta Fo_{cr}^q \approx 10^{-2}$ [7], while with a transition to the problem of reconstructing the surface temperature of the body, analogous to that considered above, the critical step for this scheme is decreased by two to three times more compared with the estimate presented for ΔFo_{cr}^q .
4. In the solution of IPHC the choice of the quantization step along the spatial coordinate must be made consistent with the value of ΔFo .

It must also be mentioned that the quantity ΔFo_{cr} increases upon the artificial introduction of fluctuational errors into the values of the function $\varphi(\tau)$. The question of the connection between the critical step and the magnitude and form of the errors in the initial data requires a separate analysis.

NOTATION

Fo , Fourier number; a , coefficient of thermal diffusivity; τ , time; $\Delta\tau$, calculated interval of time quantization; b , distance between boundary and heat sensor; T , temperature; x , spatial coordinate.

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APPROXIMATE SOLUTION OF THE LINEAR HEAT EQUATION IN HETEROGENEOUS MEDIA

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The asymptotic Vishik-Lyusternik method is used to solve the linear heat equations in heterogeneous media.

The heat equations in heterogeneous two-component continuous medium have the form [1]:

$$\begin{aligned} \varepsilon_1 \frac{\partial T_1}{\partial t} + \frac{T_1 - T_2}{\tau} = \kappa \nabla^2 T_1, \quad \frac{\partial T_2}{\partial t} + \frac{T_2 - T_1}{\tau} = \varepsilon_2 \kappa \nabla^2 T_2, \\ \varepsilon_1 = m_1 \gamma_1 c_1 / m_2 \gamma_2 c_2, \quad \varepsilon_2 = m_2 \lambda_2 / m_1 \lambda_1. \end{aligned} \quad (1)$$

Equations that are analogous to system (1) describe also the nonstationary filtration of a homogeneous liquid in crack-porous media [2, 3]. We always have the condition $\varepsilon_2 \ll 1$ [1, 4]; such a condition is not necessary for ε_1 .

In the solution of system (1), one can put $\varepsilon_2 = 0$. However, the solution of the degenerate system thus obtained can differ considerably from the exact solution of the system (1) near the boundary of the region under study. This disagreement can be avoided by using the approximate Vishik-Lyusternik method for the solution of (1) [5, 6]. We consider the method on the example of a one-dimensional problem. In this case we rewrite (1) as

$$\begin{aligned} \varepsilon_1 \frac{\partial T_1}{\partial \theta} + T_1 - T_2 = \frac{\partial^2 T_1}{\partial y^2}, \quad \frac{\partial T_2}{\partial \theta} + T_2 - T_1 = \varepsilon_2 \frac{\partial^2 T_2}{\partial y^2}, \\ \theta = t/\tau, \quad y = x/\sqrt{\kappa\tau}. \end{aligned} \quad (2)$$

The initial and boundary conditions are formulated as follows:

$$\begin{aligned} \theta = 0: \quad T_1 = T_2 = 0, \\ y = 0: \quad T_1 = T_2 = T_0 = \text{const}, \\ y \rightarrow \infty: \quad T_1 = T_2 = 0. \end{aligned} \quad (3)$$

According to the Vishik-Lyusternik method, the required solution is represented in the form of two terms

$$\tilde{T}_i = w_i + v_i, \quad i = 1, 2. \quad (4)$$

Here w_i is the solution of the degenerate system (2):

$$\varepsilon_1 \frac{\partial w_1}{\partial \theta} + w_1 - w_2 = \frac{\partial^2 w_1}{\partial y^2}, \quad \frac{\partial w_2}{\partial \theta} + w_2 - w_1 = 0 \quad (5)$$

with the following initial and boundary conditions: